

ON 3-MANIFOLDS WITH LOCALLY STANDARD $(\mathbb{Z}_2)^3$ -ACTIONS

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ABSTRACT. As a generalization of Davis-Januszkiewicz theory, there is an essential link between locally standard $(\mathbb{Z}_2)^n$ -actions (or T^n -actions) actions and nice manifolds with corners, so that a class of nicely behaved equivariant cut-and-paste operations on locally standard actions can be carried out in step on nice manifolds with corners. Based upon this, we investigate what kinds of closed manifolds admit locally standard $(\mathbb{Z}_2)^n$ -actions; especially for the 3-dimensional case. Suppose M is an orientable closed connected 3-manifold. When $H_1(M; \mathbb{Z}_2) = 0$, it is shown that M admits a locally standard $(\mathbb{Z}_2)^3$ -

action if and only if M is homeomorphic to a connected sum $\overbrace{N \# N \# \cdots \# N}^{8 \text{ copies}}$ of 8 copies of some \mathbb{Z}_2 -homology sphere N , and if further assuming M is irreducible, then M must be homeomorphic to S^3 . In addition, the argument is extended to rational homology 3-sphere M with $H_1(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and an additional assumption that the $(\mathbb{Z}_2)^3$ -action has a fixed point.

1. INTRODUCTION

During the last two decades, the theory of toric varieties has produced a strong pervasion among many mathematical areas, such as symplectic geometry, commutative algebra, toric topology etc., because toric varieties admit equivalent descriptions arising naturally in those areas. For example, see [1], [2]. In 1991, Davis and Januszkiewicz [3] introduced and studied two kinds of topological versions of toric varieties: small covers and (quasi-) toric manifolds. These two kinds of G -manifolds ($G = (\mathbb{Z}_2)^n$ or T^n) have two special properties: the group action is locally standard and the orbit space is a simple convex polytope. Generally, the local standardness of the G -action on M makes sure that the orbit space $Q = M/G$ is a nice manifold with corners. Furthermore, when Q is assumed to be a simple convex polytope P , as shown in [3], a lot of important algebraic topological invariants for M can be completely computed in terms of combinatorics of the polytope P . Indeed, let $\pi : M \rightarrow P$ be a small cover or a quasi-toric

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manifold over a simple convex polytope P . Then there is a natural map (called characteristic function) λ defined on facets of P , so that up to equivariant homeomorphism, M can be recovered from the pair (P, λ) in a canonical way (called *glue-back construction*). As a result, the topological invariants of M are also completely decided by (P, λ) .

The glue-back construction of M provides an effective way of studying small covers and quasi-toric manifolds. Following this idea, a description of topological types of all 3-dimensional small covers is given in [4] by using six kinds of cut-and-paste operations.

Moreover, it is shown in [5] that the glue-back construction can be carried out in a by far wider class than simple convex polytopes. Actually, if we only assume the G -action on M is local standard, the orbit space would be a nice manifold Q with corners which also admits a characteristic function λ when $\partial Q \neq \emptyset$. However, in this more general setting, (Q, λ) is not enough to recover the manifold M . Actually we need an additional data — a principal bundle over Q . Then, as shown in [5], up to equivariant homeomorphism M can be recovered from the characteristic function λ and a principal bundle ξ over Q , and moreover, a necessary and sufficient condition for two such manifolds over Q to be equivariantly homeomorphic is given in terms of λ and ξ .

In addition, it is well known that a general effective action may have very complicated orbit types and the structure of fixed point set, which are very difficult to be visualized. However, as a special kind of effective actions, locally standard actions can be well understood by the glue-back construction, because the structure of orbit types and fixed point set can be clearly visualized from the manifold with corner structure of the orbit space. Now, a natural question is

(\star) *What kinds of closed manifolds admit locally standard G -actions?*

The purpose of this paper is to investigate the question (\star) for $G = (\mathbb{Z}_2)^3$, so we restrict our attention to locally standard $(\mathbb{Z}_2)^3$ -actions. We shall see that although closed 3-manifolds admitting locally standard $(\mathbb{Z}_2)^3$ -actions form a wider class than 3-dimensional small covers, there are actually very few such irreducible 3-manifolds with small first \mathbb{Z}_2 -homology. We first consider \mathbb{Z}_2 -homology 3-spheres (i.e., closed connected 3-manifolds M^3 with $H_1(M^3; \mathbb{Z}_2) = 0$), and obtain the following result.

Theorem 1.1. *Suppose that M is a \mathbb{Z}_2 -homology 3-sphere. Then M admits a locally standard $(\mathbb{Z}_2)^3$ -action if and only if M is homeomorphic to a connected*
 $\overbrace{N \# N \# \cdots \# N}^{8 \text{ copies}}$ *of 8 copies of some \mathbb{Z}_2 -homology sphere N . In particular,*

if M is irreducible and admits a locally standard $(\mathbb{Z}_2)^3$ -action, then M must be homeomorphic to S^3 .

Although there are infinitely many irreducible \mathbb{Z}_2 -homology 3-spheres, Theorem 1.1 asserts that among them only S^3 admits a locally standard $(\mathbb{Z}_2)^3$ -action.

Remark 1. There is an open conjecture saying that an irreducible homotopy 3-sphere which admits an involution must be homeomorphic to S^3 (see Kirby's list [6, Problem 3.1(E)]). In our case, we actually show that an irreducible homotopy 3-sphere which allows a locally standard $(\mathbb{Z}_2)^3$ -action must be homeomorphic to S^3 . So it should be interesting to study the existence of locally standard $(\mathbb{Z}_2)^3$ -actions on a homotopy 3-sphere.

We also consider orientable rational homology 3-spheres M with $H_1(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and get the following.

Theorem 1.2. *Suppose that M is an orientable rational homology 3-sphere with $H_1(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Then M admits a locally standard $(\mathbb{Z}_2)^3$ -action having a fixed point if and only if M is homeomorphic to a connected sum $\mathbb{R}P^3 \# \overbrace{N \# N \# \cdots \# N}^{8 \text{ copies}}$ of a real projective 3-space $\mathbb{R}P^3$ and 8 copies of some \mathbb{Z}_2 -homology sphere N .*

This paper is organized as follows. In Section 2 we review some basic notions for locally standard $(\mathbb{Z}_2)^n$ -actions and manifolds with corners, and study the relationships between locally standard $(\mathbb{Z}_2)^n$ -actions and manifolds with corners. In addition, we introduce the general glue-back construction and the equivariant cut-and-paste operation. In Section 3, we list some known results on closed manifolds with a finite group action. Then we finish the proofs of our main results in Section 4.

2. LOCALLY STANDARD $(\mathbb{Z}_2)^n$ -ACTIONS AND EQUIVARIANT CUT-AND-PASTE OPERATION

First, we review some basic notions with respect to locally standard $(\mathbb{Z}_2)^n$ -actions and manifolds with corners, and then introduce some useful lemmas.

2.1. Locally standard $(\mathbb{Z}_2)^n$ -actions. A standard representation of $(\mathbb{Z}_2)^n$ on \mathbb{R}^n is the natural action defined by

$$((g_1, \dots, g_n), (x_1, \dots, x_n)) \longmapsto ((-1)^{g_1}x_1, \dots, (-1)^{g_n}x_n),$$

which exactly fixes the origin of \mathbb{R}^n and has the positive cone $\mathbb{R}_{\geq 0}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$ as its orbit space. An effective action of $(\mathbb{Z}_2)^n$ on an n -dimensional closed manifold M^n is said to be *locally standard* if it locally looks like the standard representation of $(\mathbb{Z}_2)^n$ on \mathbb{R}^n ; more precisely, if for each point x in

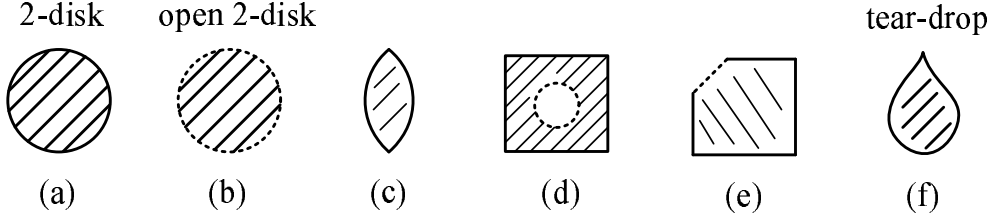


FIGURE 1. Examples of manifolds with corners

M^n , there is a $(\mathbb{Z}_2)^n$ -invariant neighborhood V_x of x such that V_x is equivariantly homeomorphic to an invariant open subset of the standard $(\mathbb{Z}_2)^n$ -representation on \mathbb{R}^n (see [3]).

Recall from [7] that an n -manifold with corners Q is a Hausdorff space together with a maximal atlas of local charts onto open subsets of $\mathbb{R}_{\geq 0}^n$ such that the overlap maps are homeomorphisms of preserving codimension, where the codimension $c(x)$ of a point $x = (x_1, \dots, x_n)$ in $\mathbb{R}_{\geq 0}^n$ is the number of x_i that is 0. Some examples of manifolds with corners are shown in Figure 1. Generally, a manifold Q with corners may be either compact or non-compact. If Q is non-compact, then Q is said to be an *open* manifold with corners. For example, given a polygon P^2 , if we remove a closed 2-disk in its interior or cut out a vertex, then the resulting space is an open 2-manifold with corners (see Figure 1 (d) and (e)).

Let Q be a manifold with corners. The *boundary* ∂Q of Q is defined as the boundary of Q as a topological manifold. Obviously, the boundary of Q is empty if and only if the codimension of each point in Q is always zero. Now suppose that Q has non-empty boundary. An *open pre-face* of Q of codimension m is a connected component of $c^{-1}(m)$. A *closed pre-face* is the closure of an open pre-face. A closed pre-face of codimension 1 is called a *facet* of Q . For any $x \in Q$, let $\Sigma(x)$ be the set of facets which contain x .

A manifold Q with corners is said to be *nice* if either ∂Q is empty or ∂Q is non-empty but $\text{Card}(\Sigma(x)) = 2$ for any x with $c(x) = 2$. It is easy to see that if an n -dimensional nice manifold Q with corners has empty boundary, then it is a closed or open topological manifold; otherwise, it has facets, and any l -dimensional closed pre-face F^l is a component of the intersection of $n - l$ facets in Q , and ∂Q is the union of all facets. The Figure 1 (f) is not nice.

A subset K of an n -manifold Q with corners is said to be an n -dimensional *open submanifold with corners* of Q if the restriction to K of the atlas of Q makes K become an open n -manifold with corners in its own right. Obviously, the intersection of $Q - K$ and the closure \bar{K} is also an $(n - 1)$ -manifold with corners, which is called the *section* of $Q - K$ and is denoted by $S(K)$.

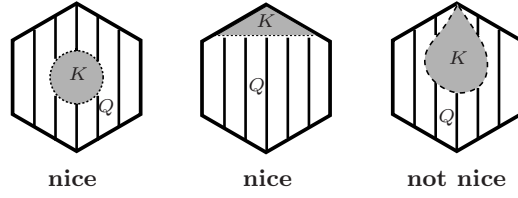


FIGURE 2. Open submanifold K with corners in a manifold Q with corners

Let Q be a nice manifold with corners, and let K be an open submanifold with corners of Q . If $S(K)$ is a nice manifold with corners, then K is said to be a *nice* open submanifold with corners of Q (see Figure 2 for examples).

A continuous map between two manifolds with corners $f : Q_1 \rightarrow Q_2$ is called a *facial map* if it preserves codimension of each point, i.e., $c(f(x)) = c(x)$ for $\forall x \in Q_1$. In particular, if f is a homeomorphism, we call it a *facial homeomorphism*.

Suppose M^n is an n -dimensional closed manifold with a locally standard $(\mathbb{Z}_2)^n$ -action. Let $\pi : M^n \rightarrow Q^n$ be the orbit map. Then Q^n is a compact nice n -dimensional manifold with corners (see also [7] and [8]). In particular, M^n is called a small cover if Q^n is a simple convex polytope (see [3]).

If the boundary of Q^n is empty, then Q^n is a closed manifold, so each open neighborhood of any point x in Q^n is identified with one in the interior of $\mathbb{R}_{\geq 0}^n$. Since the action on M^n is locally standard and M^n is compact, one has that the action on M^n must be free, so $\pi : M^n \rightarrow Q^n$ become a principal $(\mathbb{Z}_2)^n$ -bundle. Conversely, if the action on M^n is free, it is easy to see that the boundary of Q^n is empty.

Now assume that Q^n has non-empty boundary. Then ∂Q^n is the union of all its facets. Similar to the theory of small covers, each facet F of Q^n corresponds to a non-zero element $g_F \in (\mathbb{Z}_2)^n$, such that $\pi^{-1}(F)$ is fixed by the action of g_F on M^n . So a characteristic function λ on Q^n can be defined as

$$\begin{aligned} \lambda : \{\text{facets of } Q^n\} &\longrightarrow (\mathbb{Z}_2)^n \\ F &\longmapsto g_F \end{aligned}$$

satisfying the condition that whenever the intersection $\bigcap_i F_i \neq \emptyset$ of some facets is non-empty, all elements of the set $\{\lambda(F_i)\}$ are linearly independent in $(\mathbb{Z}_2)^n$. Here λ is also called a $(\mathbb{Z}_2)^n$ -coloring of Q^n .

Combining the above argument, we have

Lemma 2.1. *Suppose that M^n is a closed n -manifold admitting a locally standard $(\mathbb{Z}_2)^n$ -action, and $\pi : M^n \rightarrow Q^n$ is the orbit map. Then Q^n is a compact nice manifold with corners such that*

- (1) ∂Q^n is empty if and only if the $(\mathbb{Z}_2)^n$ -action on M^n is free (i.e., $\pi : M^n \rightarrow Q^n$ is a principal $(\mathbb{Z}_2)^n$ -bundle).
- (2) If ∂Q^n is non-empty, then Q^n admits a $(\mathbb{Z}_2)^n$ -coloring on its facets.

Remark 2. If the locally standard $(\mathbb{Z}_2)^n$ -action on M^n is not free, then it is easy to see that the boundary of Q^n together with the $(\mathbb{Z}_2)^n$ -coloring on its facets gives the information of the non-free orbits, while the interior of Q^n corresponds to all free orbits.

Lemma 2.2. *Suppose that M^n is an orientable closed connected manifold admitting a non-freely locally standard $(\mathbb{Z}_2)^n$ -action and λ is the $(\mathbb{Z}_2)^n$ -coloring on its orbit space Q^n . Then each $\tau \in \text{Im} \lambda$ is an orientation-reversing involution on M^n .*

Proof. By the definition of λ , there exists a facet F such that $\tau = \lambda(F)$. Since F is $(n-1)$ -dimensional, $\pi^{-1}(F)$ is an $(n-1)$ -dimensional closed manifold and is fixed by the involution action of τ on M^n , where $\pi : M^n \rightarrow Q^n$ is the orbit map. Since the $(\mathbb{Z}_2)^n$ -action on M^n is locally standard and M^n is orientable, $\pi^{-1}(F)$ separates M^n . So τ must be orientation-reversing. \square

Remark 3. We see from Lemma 2.1 and Lemma 2.2 that the local standardness of the action implies many special properties. Indeed, a general effective action could be much more complicated than locally standard actions. Here, we give an example of non-locally standard $(\mathbb{Z}_2)^2$ -action below which helps us to understand this difference. Consider the $(\mathbb{Z}_2)^2$ -action on the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ given by two commutative involutions τ_1 and τ_2 , where τ_1 is the reflection of S^2 about the xy -plane, τ_2 is the antipodal map. Obviously, this $(\mathbb{Z}_2)^2$ -action in a small neighborhood of the north pole and the south pole is not locally standard.

The following is a typical example for locally standard $(\mathbb{Z}_2)^n$ -actions.

Example 1. The standard linear action of $(\mathbb{Z}_2)^n$ on S^n defined by

$$((g_1, \dots, g_n), (x_0, x_1, \dots, x_n)) \longmapsto (x_0, (-1)^{g_1}x_1, \dots, (-1)^{g_n}x_n)$$

is locally standard and fixes the north pole and the south pole of S^n , but its orbit space is not a simple convex polytope except for $n = 1$. When $n = 3$, the orbit space of this standard action is the solid three-edged football \mathbf{B} , whose boundary consists of two vertices, three edges and three 2-gons.

2.2. Reconstruction of locally standard $(\mathbb{Z}_2)^n$ -actions. Suppose that M^n is a closed manifold with a locally standard $(\mathbb{Z}_2)^n$ -action, and $\pi : M^n \rightarrow Q^n$ is its orbit map. Assume Q^n is connected in the following discussions.

If the boundary of Q^n is empty, then by Lemma 2.1 the action on M^n is free, so $\pi : M^n \longrightarrow Q^n$ is actually a principal $(\mathbb{Z}_2)^n$ -bundle over Q . In other words, M^n is a 2^n -fold regular covering space over Q^n . In this case, there is no canonical way of recovering M^n from Q^n . But up to homeomorphism, M^n corresponds to an element of $H^1(Q^n; (\mathbb{Z}_2)^n)$.

If Q^n has non-empty boundary, then Q^n admits a $(\mathbb{Z}_2)^n$ -coloring λ decided by the $(\mathbb{Z}_2)^n$ action. Generally speaking, as shown in [5], the pair (Q^n, λ) is not enough for us to recover M^n . To do that, one must add another data $\xi : E \longrightarrow Q^n$, which is a principal $(\mathbb{Z}_2)^n$ -bundle over Q^n . This bundle is directly associated with M^n and can be produced in the following way: take a facet F of Q^n , one can obtain a closed submanifold $\pi^{-1}(F)$ of M^n . The required bundle is given by removing the union of small invariant tubular neighborhoods of all these $\pi^{-1}(F)$ in M^n . In particular, it is easy to see from [7] and [8] that such a bundle is uniquely determined up to isomorphism.

Conversely, from the $(\mathbb{Z}_2)^n$ -coloring λ and the principal bundle ξ over Q , we can reconstruct M^n as follows (also see [5]). First, define an equivalence \sim on E : for $x_1, x_2 \in E$,

$$x_1 \sim x_2 \iff \begin{cases} \xi(x_1) = \xi(x_2) \in \text{Int}(Q^n) = Q^n - \partial Q^n \text{ or} \\ \xi(x_1) = \xi(x_2) \in \partial Q^n \text{ and } x_1 = gx_2 \text{ for some } g \in G_F \end{cases}$$

where F is the closed pre-face of Q^n such that $\xi(x_1) = \xi(x_2)$ is contained in the relative interior of F (note that there must be some facets F_{i_1}, \dots, F_{i_r} of Q^n such that F is a component of the intersection $F_{i_1} \cap \dots \cap F_{i_r}$), and G_F is the subgroup determined by $\lambda(F_{i_1}), \dots, \lambda(F_{i_r})$. Next, up to equivariant homeomorphism, the quotient space $E / \sim = M(\lambda, \xi)$ reproduces M^n , and it is called the *glue-back construction* of M^n . In particular, since Q^n is connected and has non-empty boundary, it is easy to see that the glue-back construction $M(\lambda, \xi)$ is also connected, so is M^n . It should be pointed out that all possible locally standard $(\mathbb{Z}_2)^n$ -actions with Q^n as orbit space can be constructed in the above way.

Example 2. Let Q be the solid three-edged football \mathbf{B} in Example 1. Clearly \mathbf{B} admits a $(\mathbb{Z}_2)^3$ -coloring. Since \mathbf{B} is contractible, any principal $(\mathbb{Z}_2)^3$ -bundle over \mathbf{B} is always trivial. Thus, given a $(\mathbb{Z}_2)^3$ -coloring λ on \mathbf{B} , up to equivariant homeomorphism we can always construct a unique closed 3-manifold with a locally standard $(\mathbb{Z}_2)^3$ -action such that its orbit space is \mathbf{B} . Furthermore, it is easy to check that any closed 3-manifold with a locally standard $(\mathbb{Z}_2)^3$ -action whose orbit space is \mathbf{B} is weakly equivariantly homeomorphic to S^3 with the standard $(\mathbb{Z}_2)^3$ -action.

2.3. Equivariant cut-and-paste operation. Suppose that M_1^n and M_2^n are two closed manifolds with non-freely locally standard $(\mathbb{Z}_2)^n$ -actions, and Q_1 and Q_2 are their corresponding orbit spaces with $(\mathbb{Z}_2)^n$ -colorings λ_1 and λ_2 respectively. A facial map $f : Q_1 \rightarrow Q_2$ is said to be *weak color-matching* if there is an automorphism $\sigma : (\mathbb{Z}_2)^n \rightarrow (\mathbb{Z}_2)^n$ such that $\sigma(\lambda_2(f(F))) = \lambda_1(F)$ for any facet F of Q_1 . In particular, if $\lambda_2(f(F)) = \lambda_1(F)$ for any facet F of Q_1 , then f is said to be *color-matching*.

It is shown in [4] that all 3-dimensional small covers can be constructed from some basic ones using six equivariant cut-and-paste operations. The common idea of these operations is to glue two manifolds with locally standard $(\mathbb{Z}_2)^n$ -actions together along a family of $(\mathbb{Z}_2)^n$ -orbits of the same type in an equivariant way. Using above definitions, we can make this more precise.

Suppose that K_1 and K_2 are nice open submanifolds with corners of Q_1 and Q_2 respectively such that $\pi_1^{-1}(K_1)$ is equivariant homeomorphic to $\pi_2^{-1}(K_2)$. Note that by the glue-back construction, $\pi_i^{-1}(K_i)$ is an open submanifold of M_i^n ($i = 1, 2$). Then we can construct a closed manifold $M^\#$ with a locally standard $(\mathbb{Z}_2)^n$ -action as follows:

Let \tilde{g} be the equivariant homeomorphism from $\pi_1^{-1}(K_1)$ to $\pi_2^{-1}(K_2)$. Obviously, \tilde{g} determines an equivariant homeomorphism \tilde{g}_1 from the boundary of $M_1 - \pi_1^{-1}(K_1)$ to that of $M_2 - \pi_2^{-1}(K_2)$. In the orbit spaces, \tilde{g} induces a natural homeomorphism $g : K_1 \rightarrow K_2$ and \tilde{g}_1 determines a homeomorphism $g_1 : S(K_1) \rightarrow S(K_2)$. In particular, if K_1 and K_2 have non-empty boundaries, then their facets naturally inherit the colorings from Q_1 and Q_2 , and g is a color-matching facial homeomorphism in this case. If $S(K_1)$ and $S(K_2)$ have non-empty boundaries, then g_1 is also a color-matching facial homeomorphism. Now, the required manifold $M^\#$ is defined as

$$M^\# = (M_1 - \pi_1^{-1}(K_1)) \cup_{\tilde{g}_1} (M_2 - \pi_2^{-1}(K_2)),$$

which is obtained by gluing $M_1 - \pi_1^{-1}(K_1)$ and $M_2 - \pi_2^{-1}(K_2)$ along their boundaries via \tilde{g}_1 . $M^\#$ naturally admits a locally standard $(\mathbb{Z}_2)^n$ -action. Correspondingly, its orbit space $Q^\# = M^\# / (\mathbb{Z}_2)^n$ is

$$Q^\# = (Q_1 - K_1) \cup_{g_1} (Q_2 - K_2),$$

which is obtained from $Q_1 - K_1$ and $Q_2 - K_2$ by gluing $S(K_1)$ and $S(K_2)$ via g_1 (see Figure 3). In addition, the manifold with corners structure and the $(\mathbb{Z}_2)^n$ -coloring on $Q^\#$ is defined by:

- (a) when $S(K_1)$ and $S(K_2)$ have non-empty boundaries, for each m -dimensional pre-face f_1^m of $S(K_1)$, let $f_2^m = g(f_1^m) \subset S(K_2)$. Then there is a unique $(m+1)$ -dimensional closed pre-face $F_1^{m+1} \subset Q_1 - K_1$ and $F_2^{m+1} \subset Q_2 - K_2$ such that $f_1^m \subset \partial F_1^{m+1}$ and $f_2^m \subset \partial F_2^{m+1}$. Then F_1^{m+1} and F_2^{m+1} is glued

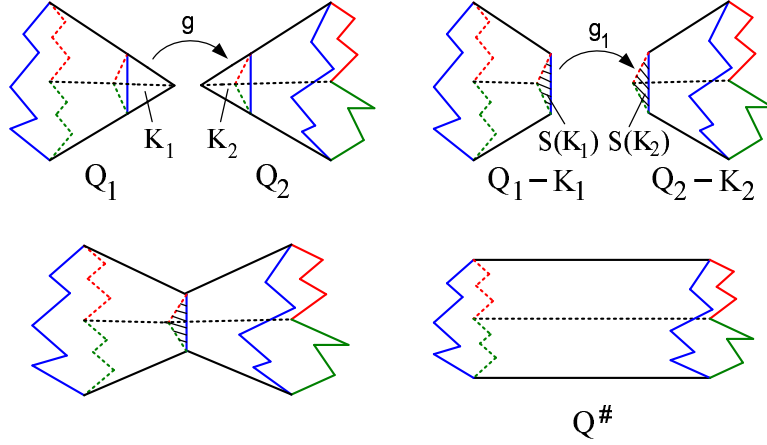


FIGURE 3. Cut-and-Paste operation on orbit spaces

together and define a closed pre-face $F^{m+1} = F_1^{m+1} \cup_{g_1} F_2^{m+1}$ in $Q^\#$ where the sectional face f_i^m is removed (see Figure 3). Clearly, $Q^\#$ is also a nice manifold with corners since $S(K_1)$ and $S(K_2)$ are nice manifolds with corners. The $(\mathbb{Z}_2)^n$ -coloring $\lambda^\#$ on $Q^\#$ can be given in the following way: given a facet F of $Q^\#$, if F is a facet of $Q_1 - K_1$ (or $Q_2 - K_2$), then $\lambda^\#(F)$ is defined as $\lambda_1(F)$ (or $\lambda_2(F)$). If F is the facet determined by two facets $F_1 \subset Q_1 - K_1$ and $F_2 \subset Q_2 - K_2$, then we define $\lambda^\#(F) = \lambda_1(F_1) = \lambda_2(F_2)$.

- (b) when $S(K_1)$ and $S(K_2)$ have empty boundaries, they are in the interiors of Q_1 and Q_2 respectively. In this case, gluing $(Q_1 - K_1)$ and $(Q_2 - K_2)$ together via g_1 does not touch the corner structures of Q_1 and Q_2 . Then each facet of $Q^\#$ belongs exactly to one of Q_1 or Q_2 , and the $(\mathbb{Z}_2)^n$ -coloring $\lambda^\#$ on $Q^\#$ is naturally defined by λ_1 and λ_2 .

We denote the above procedure by:

$$(M^\#, Q^\#, \lambda^\#) = (M_1, Q_1, \lambda_1) \#_{K_1, K_2} (M_2, Q_2, \lambda_2),$$

and call $(M^\#, Q^\#, \lambda^\#)$ the *equivariant cut-and-paste operation* of (M_1, Q_1, λ_1) and (M_2, Q_2, λ_2) via K_1 and K_2 .

As an example, the equivariant connected sum is a special equivariant cut-and-paste operation.

Remark 4. Using the glue-back construction, we can use the equivariant cut-and-paste operations on the level of orbit spaces to understand the corresponding equivariant operations on manifolds.

3. ORIENTATION-REVERSING INVOLUTION ON 3-MANIFOLDS

Recall that a connected closed 3-manifold M is called *prime* if $M \approx P \# Q$ implies $P \approx S^3$ or $Q \approx S^3$. M is called *irreducible* if every embedded 2-sphere in M bounds a 3-ball in M . In particular, if M is not a S^2 -bundle over S^1 , then M is prime if and only if it is irreducible. It is well known that any orientable connected closed 3-manifold M is a connected sum of finitely many prime 3-manifolds, and each connected summand is uniquely determined by M up to homeomorphism.

It is well known that not all closed 3-manifolds admit an involution. Indeed, it is shown in [9] that there are infinitely many irreducible homology 3-spheres that do not admit any PL -involutions.

On the other hand, if a closed 3-manifold M is orientable and admits an orientation reversing involution σ , then the fixed point set of σ has some topological restrictions from the homology group of M .

Theorem 3.1 (Kobayashi [10]). *For a closed orientable 3-manifold M which admits an orientation-reversing involution τ ,*

$$\dim_{\mathbb{Z}_2} H_1(\text{Fix}(\tau); \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) + \beta_1(M).$$

where $\beta_1(M)$ is the first Betti number of M .

It is possible that an involution on a 3-manifold has a disconnected fixed point set. For example, the fixed point set of the obvious involution on $\mathbb{R}P^3$ via reflection is the union of an $\mathbb{R}P^2$ and a point.

In addition, let us recall some famous theorems on finite group actions on manifolds which will be useful in the proofs of our main results in section 4.

Theorem 3.2 (Borel [11]). *Suppose that $G = (\mathbb{Z}_p)^k$ acts on an n -dimensional \mathbb{Z}_p -homology sphere X . For any subgroup H of G , the fixed point set X^H is a \mathbb{Z}_p -homology sphere of dimension, say, $n(H)$. Then*

$$n - n(G) = \sum_H (n(H) - n(G)),$$

where the sum is over all H of corank 1.

Theorem 3.3 (tom Dieck [12] and Huang [13]). *For a periodic transformation f on a compact smooth manifold, $L(f) = \chi(\text{Fix}(f))$, where $L(f)$ is the Lefschetz number of f .*

4. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

The objective of this section is to give proofs of our main results. First, let us prove Theorem 1.1.

Proof of Theorem 1.1. With the assumptions on M , M must be orientable because $H_1(M, \mathbb{Z}_2) = 0$. Suppose that M admits a locally-standard $(\mathbb{Z}_2)^3$ -action. Let $\pi : M \rightarrow Q$ be the orbit map. Borel's Theorem 3.2 implies that the $(\mathbb{Z}_2)^3$ -action on M can not be a free action (also see [14] for more general cases), so by Lemma 2.1, Q has non-empty boundary and admits a $(\mathbb{Z}_2)^3$ -coloring λ . Furthermore, by Lemma 2.2, each $\tau \in \text{Im}\lambda$ is an orientation-reversing involution on M such that its fixed point set $\text{Fix}(\tau)$ has 2-dimensional components. In addition, by Theorem 3.2, we have $H_1(\text{Fix}(\tau); \mathbb{Z}_2) = 0$, so the 2-dimensional components in the $\text{Fix}(\tau)$ must be 2-spheres (note that this can also be seen from the Smith theory). Moreover, by Theorem 3.3 and the universal coefficient theorem, $\chi(\text{Fix}(\tau)) = 2$. This implies that $\text{Fix}(\tau)$ contains only one 2-dimensional component, which is exactly one S^2 .

Let $\langle \tau \rangle$ denote the \mathbb{Z}_2 subgroup generated by τ . Since the $(\mathbb{Z}_2)^3$ -action on M is locally standard, the action of $(\mathbb{Z}_2)^3 / \langle \tau \rangle$ on the unique 2-dimensional component (i.e., S^2) of $\text{Fix}(\tau)$ is still locally standard. Therefore, the boundary of Q is tessellated by patches that are orbit spaces of some S^2 's with locally standard $(\mathbb{Z}_2)^2$ -actions.

Claim 1. *The boundary of Q is tessellated by 2-gons.*

In fact, if Σ is a closed surface with a locally-standard $(\mathbb{Z}_2)^2$ -action, its orbit space F is a 2-manifold with corners and ∂F is not empty. We know from [5, Corollary 4.2] that

$$\chi(\Sigma) = 4\chi(F) - m, \quad (1)$$

where m is the number of vertices in F (i.e., the number of the points fixed by the $(\mathbb{Z}_2)^2$ -action).

In particular when $\Sigma = S^2$, $\chi(\Sigma) = 2$. Since $\chi(F) \leq 1$, so $\chi(F) = 1$ since ∂F is not empty. Then $m = 2$ and F is homeomorphic to the 2-disk D^2 , so F must be a 2-gon.

Thus, topologically the boundary ∂Q must consist of some 2-spheres, each of which is covered by 2-gon facets such that each vertex of Q meets exactly 3 facets. So each boundary component of Q is actually a union of exactly three 2-gons, as shown in Figure 4.

Claim 2. *∂Q is connected.*

Suppose that Q has k boundary components (2-spheres). If $k > 1$, then we can do an equivariant cut-and-paste operation between (M, Q, λ) and $(S^3, \mathbf{B}, \lambda_0)$,

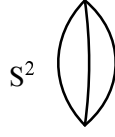


FIGURE 4. 2-sphere covered by 2-gons

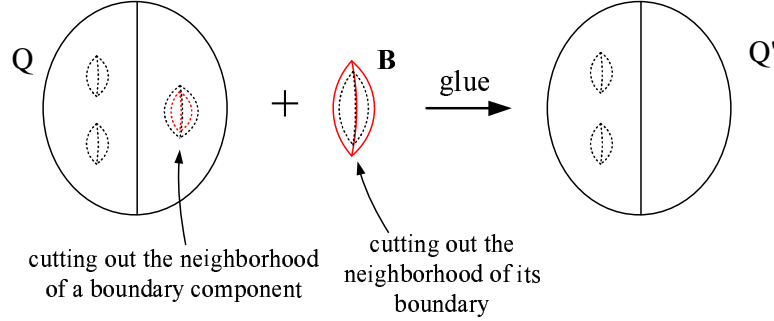


FIGURE 5. Fill a hole by a 3-ball

which is stated in terms of orbit spaces as follows: cut out an open neighborhood $N(C)$ of a boundary component C of Q and an open neighborhood $N(\partial \mathbf{B})$ of the boundary component of the solid three-edged football \mathbf{B} , and then glue them together along the new boundaries, as shown in Figure 5. Write $(M', Q', \lambda') = (M, Q, \lambda) \#_{N(C), N(\partial \mathbf{B})} (S^3, \mathbf{B}, \lambda_0)$. On the other hand, this operation is invertible with respect to (M, Q, λ) and (M', Q', λ') ; in other words, we can cut out a small open 3-ball B from the interior of Q' and a solid three-edged football \mathbf{B} respectively, and then glue them along the new boundaries to get Q , as shown in Figure 6. Thus, $(M, Q, \lambda) = (M', Q', \lambda') \#_{N(C), N(\partial \mathbf{B})}^{-1} (S^3, \mathbf{B}, \lambda_0)$, where $\#_{N(C), N(\partial \mathbf{B})}^{-1} = \#_{B, B}$ is the equivariant connected sum along free orbits. Since $k - 1 > 0$, $\partial Q'$ still contains some 2-spheres as in Figure 6, so M' is still a connected 3-manifold. We see that M is actually obtained by attaching a connected 3-manifold to S^3 by eight disjoint $S^2 \times I$ tubes (see Figure 7). This implies that $\text{rank} H_1(M, \mathbb{Z}) \geq 7$, which contradicts $H_1(M; \mathbb{Z}_2) = 0$. Thus, k must be 1, so ∂Q is connected.

Now we know that $\partial Q \approx S^2$. We can still carry out the operation

$$(M, Q, \lambda) \#_{N(S^2), N(\partial \mathbf{B})} (S^3, \mathbf{B}, \lambda_0)$$

as above, and still denote it by (M', Q', λ') . Then it is easy to see that Q' becomes a closed manifold! In this case, all the orbits of the $(\mathbb{Z}_2)^3$ -action on M' are free orbits. But M' may not be connected. Notice that each connected component of M' must be a covering space of Q' , whose covering cardinality divides 8. By the

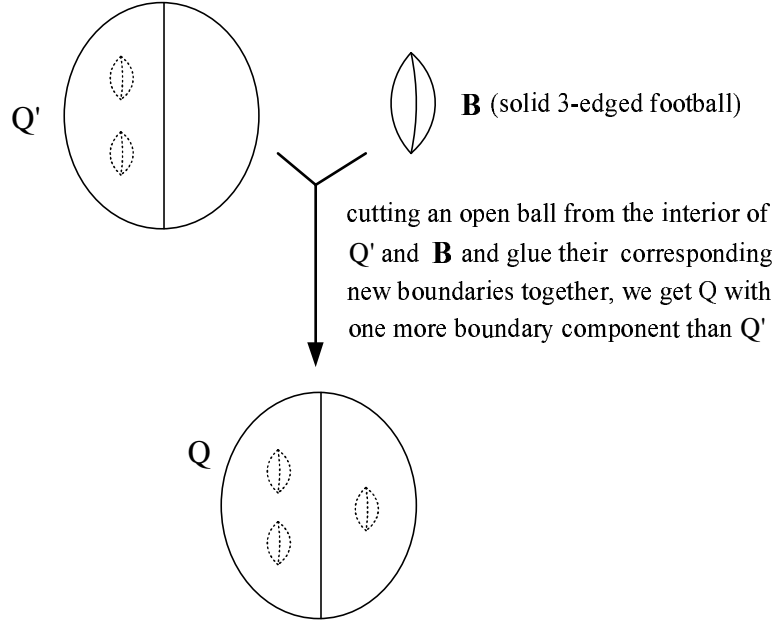


FIGURE 6.

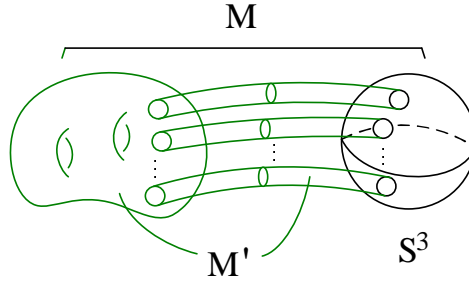


FIGURE 7.

equivariance of the $(\mathbb{Z}_2)^3$ -action, M' must have 1, 2, 4 or 8 connected components. Topologically, the possible relations between M and M' are shown in Figure 7, Figure 8 and Figure 9.

The cases listed in Figure 7 and Figure 8 are not possible because $H_1(M; \mathbb{Z}_2) = 0$. So the only possible case is Figure 9 where M is the connected sum of eight copies of Q' with a S^3 . This implies that Q' is a \mathbb{Z}_2 -homology sphere. So M is

of the form $M = \overbrace{N \# N \# \cdots \# N}^{8 \text{ copies}}$, where N is a \mathbb{Z}_2 -homology sphere. Conversely, a closed 3-manifold M of this form obviously admits a locally-standard $(\mathbb{Z}_2)^3$ -action.

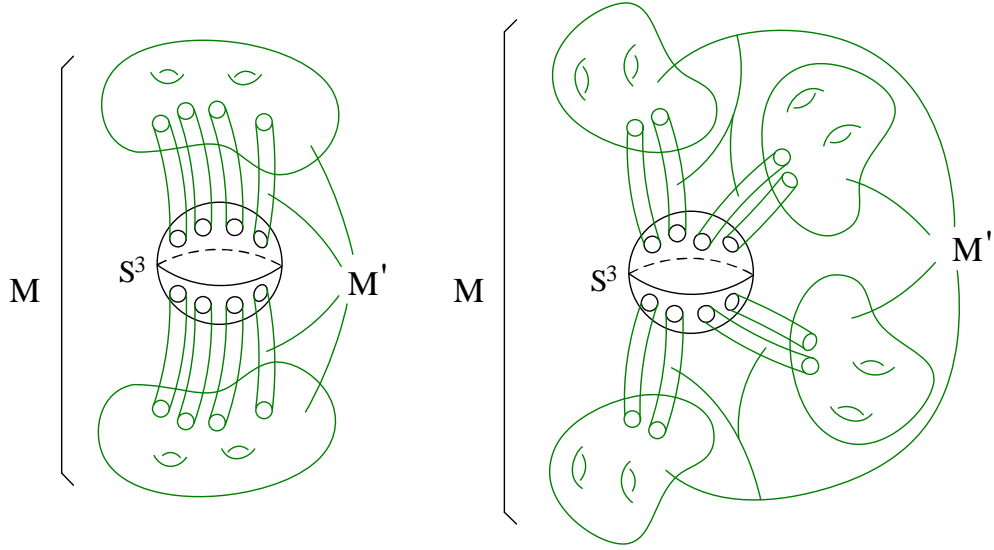
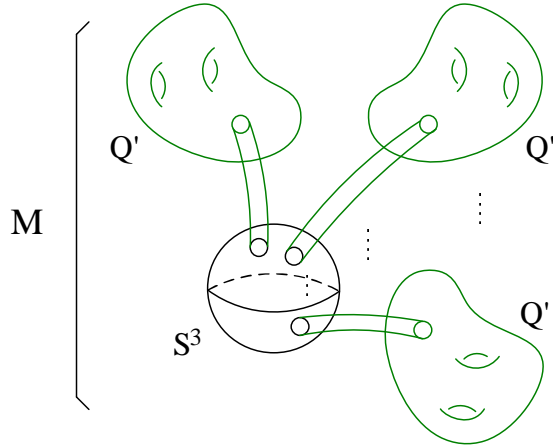


FIGURE 8.

FIGURE 9. Eight Q' -handles attached to S^3

If we assume M is irreducible, then Q' in Figure 9 must be homeomorphic to S^3 . So Q is homeomorphic to a 3-ball whose boundary consists of 2-gon facets. Then Q must be the solid three-edged football **B**. Hence M must be S^3 with a locally standard $(\mathbb{Z}_2)^3$ -action. \square

Now let us complete the proof of Theorem 1.2.

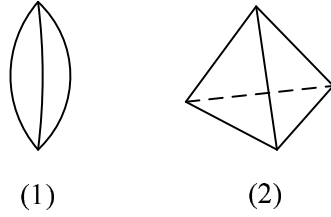


FIGURE 10.

Proof of Theorem 1.2. With the conditions on M , if M admits a locally standard $(\mathbb{Z}_2)^3$ -action having a fixed point, then the orbit space Q of the action has non-empty boundary, so Q admits a $(\mathbb{Z}_2)^3$ -coloring λ . Furthermore, for each $\tau \in \text{Im}\lambda$, by Lemma 2.2, τ is an orientation-reversing involution on M such that its fixed point set $\text{Fix}(\tau)$ has 2-dimensional components. By Theorem 3.1,

$$\dim_{\mathbb{Z}_2} H_1(\text{Fix}(\tau); \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) + \beta_1(M) \leq 1.$$

So the fixed point set of τ must be the disjoint union of 2-spheres, or real projective planes, or discrete points. Theorem 3.3 implies that $\text{Fix}(\tau)$ is either a S^2 , a $\mathbb{R}P^2$ plus a point or two points. By the equation (1), the orbit space of $\mathbb{R}P^2$ under a locally-standard $(\mathbb{Z}_2)^2$ -action can only be a triangle whose vertices are fixed points. So the boundary of Q must consist of 2-spheres, because no other closed surfaces can be covered by 2-gons and 3-gons with each vertices incident to three different facets (again an easy calculation of the Euler number). The only possible tessellations of each boundary component of Q in this case are (see Figure 10):

- (1) three 2-gons;
- (2) four triangles as the boundary of a regular tetrahedron.

As the orbit spaces of locally-standard $(\mathbb{Z}_2)^3$ -actions, solid three-edged football must correspond to S^3 and tetrahedron must correspond to $\mathbb{R}P^3$.

Assume the orbit space Q has k boundary components. Similarly to the proof of Theorem 1.1, we can show that $k = 1$ using the condition $H_1(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

If assuming M is irreducible, Q must be a 3-ball whose boundary is one of the above two cases. (The only new ingredient here is that the “ S^3 ” in the Figure 7, Figure 8 and Figure 9 above could be S^3 or $\mathbb{R}P^3$ in this case). Considering the homology, M must be $\mathbb{R}P^3$. But without the “irreducibility” condition of M , M could be homeomorphic to

$$\mathbb{R}P^3 \# \overbrace{N \# N \# \cdots \# N}^{8 \text{ copies}}$$

where N is a \mathbb{Z}_2 -homology 3-sphere.

Conversely, if M is homeomorphic to the connected sum $\mathbb{R}P^3 \# \overbrace{N \# N \# \cdots \# N}^{8 \text{ copies}}$ with N a \mathbb{Z}_2 -homology 3-sphere, then obviously M admits a locally standard $(\mathbb{Z}_2)^3$ -action such that the fixed point set is non-empty. \square

Remark 5. General effective $(\mathbb{Z}_2)^n$ -actions on cohomology projective spaces were investigated in [16], and some general structural results of these actions were obtained there.

Remark 6. We might be able to remove the extra condition in Theorem 1.2 that the locally-standard $(\mathbb{Z}_2)^3$ -action has a fixed point to arrive at the same conclusion. But we must show that there are no free $(\mathbb{Z}_2)^3$ -actions on a rational homology sphere M with $H_1(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Question: Is there any free $(\mathbb{Z}_2)^3$ -action on a rational homology 3-sphere M with $H_1(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$?

The arguments in above theorems could be extended to investigate general 3-manifolds with locally-standard $(\mathbb{Z}_2)^3$ -actions. The main difficulty in the general cases is that the boundary components of the orbit space of the action might have closed surfaces with higher genus, which make this approach very complicated.

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